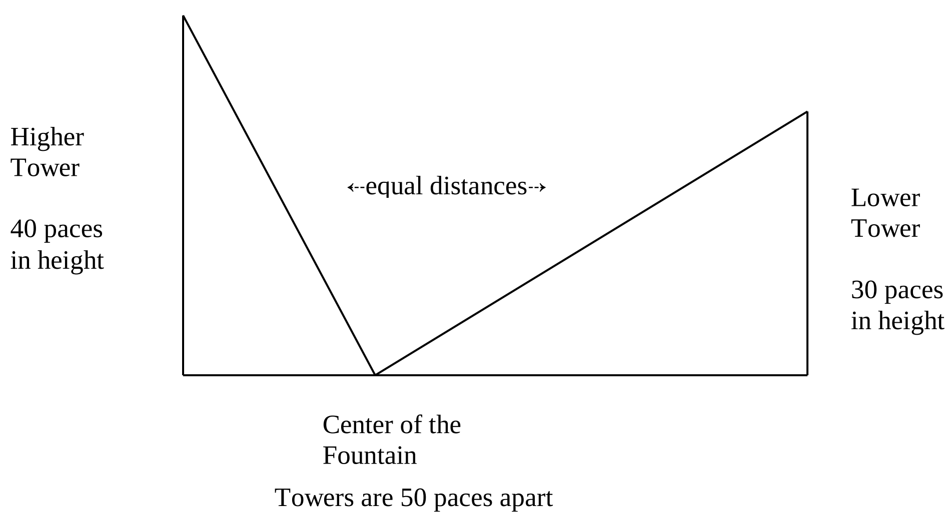
A Transcription and Explication using Modern English and Notation

From: Leonardo Pisano (Fibonacci) [c.1170-c.1240], born in Pisa, in (what is now) Italy, from *Liber Abaci*, (Book of Calculation), 1202, Chapter 13, Part One, as translated by L. E. Sigler, 2002, pp. 462-463.

Notes: All comments in smaller font and [square brackets] are mine.

On Two Birds

Two birds were above the height of two [assumed vertical] towers; one tower was 40 paces in height and the other 30, and they [the towers] were 50 paces apart; at an instant the pair of birds descended [from the tops of the towers] flying to the center [This refers to some point between the towers; but not the exact center.] where there was a fountain, and they arrived at the same moment at the fountain which was between both towers. From the moment they left until the moment they arrived they flew in straight lines from the tops of the towers to the center of the fountain; the flights were of equal lengths [distances];



in geometry it is clearly demonstrated that the height of either tower multiplied by itself added to the distance from the tower to the center of the fountain multiplied by itself is the same as the straight line from the center of the fountain to the top of the tower multiplied by itself [This is a statement of the Pythagorean Theorem, Euclid I-47.];

this therefore known, you put it that the distance from the center of the fountain to the higher tower is any number of paces, we say 10, [Here he begins an application of the "Method of Double False Position". We'll discuss below whether this is appropriate.] and you multiply the 10 by itself; there will be

100 that you add to the height of the higher tower multiplied by itself, namely to 1600; there will be 1700 that you keep, and you multiply by itself the remaining distance, namely the 40 which is the distance from the center [of the fountain] to the lower tower; there will be 1600 which you add to the height of the lower tower multiplied by itself, namely 900; this makes 2500 that should be 1700 as was the sum of the other two products [by the Pythagorean Theorem]; therefore [the value of] this position is long of the true value by 800, namely the difference between 1700 and 2500 [The "value of this position" is really the difference "lower tower sum of squares – higher tower sum of squares" which is a function of the "position" (which here is 10 paces), and the "true value" is zero!];

therefore you lengthen the distance from the center of the fountain to the higher tower; indeed it is lengthened 5 paces from the first position, namely 15 paces from the center [of the fountain] to the higher tower, and you multiply the 15 by itself; there will be 225 which you add to height of the higher tower multiplied by itself, namely 1600; there will be 1825. Similarly you multiply by itself the 35 which is the distance from the center of the fountain to the lower tower making 1225 [typographic error corrected from "12225"]; this added to the 900, namely the height of the higher [should be "lower"] tower multiplied by itself, makes 2125 that should be 1825 by the abovewritten rule [meaning the Pythagorean Theorem]. Therefore the value of the second position is an amount long of the true value by 300 [Once again, the "value of the second position" is really "lower tower sum of squares – higher tower sum of squares" which is a function of the "position" (here, 15 paces), and the "true value" is zero!];

the first value was long indeed by 800; therefore you say:
for the five paces which we lengthened the distance from the center of the fountain to the higher tower we approximated more closely to the true value by 500; how much indeed shall we lengthen the distance from the center of the fountain to the same higher tower in order to improve the approximation by 300?

You multiply the 5 by the 300, and you divide by 500; the quotient will be 3 paces [written within a small box below]

500		5
	*	
	*	
300		3

[The idea is that when $\frac{5}{500}=\frac{\Delta}{300}$, it follows that $\frac{5\cdot 300}{500}=\Delta$. The large box above is meant to suggest, reading horizontally, that since a change of 500 "was caused by" an increase of 5 paces, then a change of 300 "will be caused by" an increase of 3 paces. The first column can be thought of as a function of the second column.]

which added to the 15 paces yields 18 paces, and this will be the distance from the [center of the] fountain to the higher tower. Truly the remaining distance, namely the 32 [which is 50 - 18], is the distance to the lower tower.

[Finally, he checks the solution.]

For example, the product of the 18 by itself added to the product of the 40 by itself makes as much as the product of the 32 by itself added to the product of the 30 by itself, as had to be.

[$18^2 + 40^2 = 32^2 + 30^2$ (= 1924)]

A Modern Direct Solution

Let x = the distance from the higher tower to the center of the fountain, so that
$50 - x$ = the distance from the lower tower to the center of the fountain.

The problem requires that [Higher tower sum of squares] = [Lower tower sum of squares], so:

$$x^2 + 40^2 = (50 - x)^2 + 30^2$$
$$x^2 + 40^2 = 50^2 - 100x + x^2 + 30^2$$
$$100x + 40^2 - 50^2 - 30^2 = 0$$
$$100x - 2 \cdot 30^2 = 0$$
$$100(x - 18) = 0$$
$$x = 18$$

So x = 18 paces is the solution. Notice that it becomes clear at the third equation that in fact this is a linear equation (the "squared" powers add to zero), justifying the use of the Method of Double False Position mentioned above, which only applies to linear relationships (functions). [Observe that the original choice of lengths is computationally convenient since $30^2 + 40^2 = 50^2$.]

A Modern Interpretation of the Leonardo Pisano Solution

Using x and $50 - x$ as above, he examines the function of x (which he calls "the value of the position") given by

$f(x)$ = [Lower tower sum of squares] - [Higher tower sum of squares]

meaning $f(x) = \left[(50 - x)^2 + 30^2\right] - \left[x^2 + 40^2\right]$

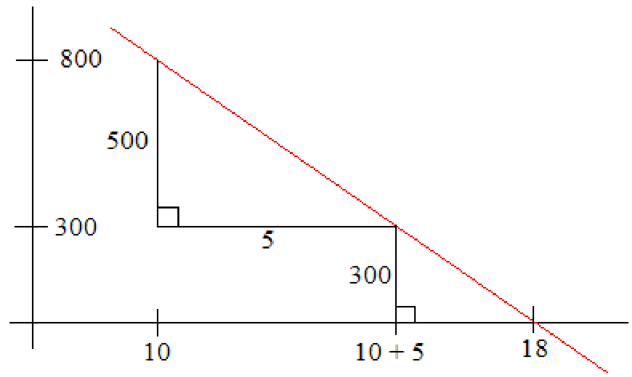
which he desires to equal zero (the "true value"), that is, $f(x) = 0$.

Using the Method of Double False Position, which as we noted above does apply here since the function is actually linear, he first calculates

$f(10) = 800$

then $f(10 + 5) = 300 = 800 - 500$

and concludes that if an increase of x by 5 yields a decrease of $f(x)$ by 500, then the final decrease by 300 down to zero will follow from a further increase of x by 3. Thus $x = 15 + 3 = 18$ paces is the solution.


800
500
300
5
300
10
10 + 5
18